Type inference for monotonicity

Michael Arntzenius

University of Birmingham

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$$x + \log x$$

$$-x$$

$$x - \log x$$

$$4$$

$$
\begin{array}{lll}
\text{types} \quad A, B & ::= & \mathbb{R} \mid A \to B \\
\text{terms} \quad M, N & ::= & x \mid \lambda x.\, M \mid M\ N
\end{array}
$$

$$
\begin{array}{rcl}
[\![A]\!] & \in & \textbf{Poset} \\
[\![\mathbb{R}]\!] & = & \mathbb{R} \\
[\![A \to B]\!] & = & \textit{monotone}\ \text{maps}\ [\![A]\!] \to [\![B]\!], \\
 & & \text{ordered pointwise}
\end{array}
$$

$$
\begin{aligned}
\lambda x.\, x + \log x \quad &: \quad \mathbb{R} \to \mathbb{R} \\
\lambda x.\, -x \quad &: \quad ???? \to \mathbb{R} \\
\lambda x.\, x - \log x \quad &: \quad ??? \to \mathbb{R} \\
\lambda x.\, 4 \quad &: \quad ??? \to \mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
\lambda x.\, x + \log x &\quad : \quad \mathbb{R} \to \mathbb{R} \\
\lambda x.\, -x &\quad : \quad \mathrm{op}\,\mathbb{R} \to \mathbb{R} \\
\lambda x.\, x - \log x &\quad : \quad \Box\mathbb{R} \to \mathbb{R} \\
\lambda x.\, 4 &\quad : \quad \Diamond\mathbb{R} \to \mathbb{R}
\end{aligned}
$$

Let $\Box A$ be A, ordered *discretely*:

$$x \leqslant y : \Box A \iff x = y : A$$

Then $f : \Box A \to B$ is monotone iff

$$x = y \implies f(x) \leqslant f(y)$$

i.e. **always**!

$$\begin{aligned}
\lambda x.\, x + \log x &\quad : \quad \mathbb{R} \to \mathbb{R} \\
\lambda x.\, -x &\quad : \quad \mathrm{op}\,\mathbb{R} \to \mathbb{R} \\
\lambda x.\, x - \log x &\quad : \quad \square\mathbb{R} \to \mathbb{R} \\
\lambda x.\, 4 &\quad : \quad \lozenge\mathbb{R} \to \mathbb{R}
\end{aligned}$$

$\lozenge A$ *identifies* weakly connected elements in A:

$$x \leqslant y \vee y \leqslant x : A \implies x = y : \lozenge A$$

$\Diamond A$ *identifies* weakly connected elements in A:

$$x \leqslant y \vee y \leqslant x : A \implies x = y : \Diamond A$$

Theorem:

$$f : \Diamond A \to B \iff f : A \to \Box B$$

$$
\begin{array}{lcl}
\lambda x.\, x + \log x & : & \mathbb{R} \to \mathbb{R} \\
\lambda x.\, -x & : & \mathrm{op}\, \mathbb{R} \to \mathbb{R} \\
\lambda x.\, x - \log x & : & \Box\mathbb{R} \to \mathbb{R} \\
\lambda x.\, 4 & : & \Diamond\mathbb{R} \to \mathbb{R}
\end{array}
$$

$\Box$ is a *necessity modality* / *monoidal comonad*;
$\Diamond$ is a *possibility modality* / *monoidal monad*.

Pfenning & Davies gave typing rules for these in *A Judgmental Reconstruction of Modal Logic*!

$$(\lambda x.\, x - \log x) : \Box\mathbb{R} \to \mathbb{R}$$

$$(\lambda x.\ x - \log x) : \Box\mathbb{R} \to \mathbb{R}$$

$$(\lambda x.\ \textbf{let box}\ u = x\ \textbf{in}\ u - \textbf{box}\,(\log u)) : \Box\mathbb{R} \to$$

: (

How to infer monotonicity:

1. Infer variable usage modes bottom-up.
2. Use modal subtyping for implicit coercion.

1. Bottom-up mode inference

$$\Gamma \vdash M : B$$

1. Bottom-up mode inference

$$x : [\mathrm{T}]\, A \vdash M : \mathrm{B}$$

1. Bottom-up mode inference

$$x : [T]\, A \vdash M : B$$

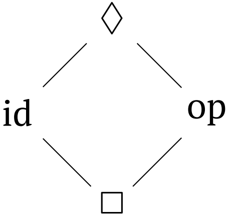
$$\text{modes} \quad T \quad ::= \quad \text{id} \mid \text{op} \mid \Box \mid \Diamond$$

1. Bottom-up mode inference: duplication

$$\frac{x : [T]\, A \vdash M : A \qquad x : [U]\, A \vdash N : B}{x : [??????]\, A \vdash (M, N) : A \times B}$$

1. Bottom-up mode inference: duplication

$$\frac{x : [T]\,A \vdash M : A \qquad x : [U]\,A \vdash N : B}{x : [T \wedge U]\,A \vdash (M, N) : A \times B}$$



1. Bottom-up mode inference: composition

$$\frac{x : [T]\, A \vdash M : B \qquad y : [U]\, B \vdash N : C}{x : [UT]\, A \vdash \mathbf{let}\ y = M\ \mathbf{in}\ N}$$

1. Bottom-up mode inference: composition

$$\frac{x : [T]\, A \vdash M : B \qquad y : [U]\, B \vdash N : C}{x : [UT]\, A \vdash \mathbf{let}\; y = M \;\mathbf{in}\; N}$$

1. Bottom-up mode inference: composition

$$\frac{x : [T]\, A \vdash M : B \qquad y : [U]\, B \vdash N : C}{x : [UT]\, A \vdash \mathbf{let}\ y = M\ \mathbf{in}\ N}$$

Composition UT is a monoid, with id neutral:

UT		T: id	op	$\Box$	$\Diamond$
U	id	id	op	$\Box$	$\Diamond$
	op	op	id	$\Box$	$\Diamond$
	$\Box$	$\Box$	$\Box$	$\Box$	$\Diamond$
	$\Diamond$	$\Diamond$	$\Diamond$	$\Box$	$\Diamond$

Composition UT and meet $T \wedge U$ form a semiring!

Other systems with semiring-valued annotations:

1. *Linear Haskell* (POPL 2018)
2. *I Got Plenty o' Nuttin* (McBride)
3. *Monotonicity Types* (PMLDC 2017)
4. *Bounded Linear Types in a Resource Semiring* (ESOP 2014)
5. *The Next 700 Modal Type Assignment Systems* (TYPES 2015)
6. *A Fibrational Framework for Substructural and Modal Logics* (FSCD 2017) **massively** generalizes this pattern.

... more?

2. Modal subtyping

$$(\lambda x.\, \textbf{let box}\, u = x\, \textbf{in}\, u - \textbf{box}\,(\log u)) : \Box\mathbb{R} \to$$

2. Modal subtyping: □-introduction?

SUBSUMPTION

$$\frac{\Gamma \vdash M : A \qquad A <: B}{\Gamma \vdash M : B}$$

But $A \not<: \Box A$!

2. Modal subtyping!

$$[\mathsf{U}]\, A <: B$$

Finds the *most general* mode U such that $\mathsf{U}A <: B$.

2. Modal subtyping: □-introduction!

SUBSUMPTION

$$\frac{x : [T]\,A \vdash M : B \qquad [U]\,B <: C}{x : [UT]\,A \vdash M : C}$$

2. Modal subtyping: □-introduction!

SUBSUMPTION

$$\frac{x : [T]\,A \vdash M : B \qquad [U]\,B <: C}{x : [UT]\,A \vdash M : C}$$

SUBTYPING INTO □

$$\frac{[T]\,A <: B}{[\Box T]\,A <: \Box B}$$

2. Modal subtyping: □-introduction!

SUBSUMPTION

$$\frac{x : [T]\, A \vdash M : B \qquad [U]\, B <: C}{x : [UT]\, A \vdash M : C}$$

SUBTYPING INTO □

$$\frac{[T]\, A <: B}{[\Box T]\, A <: \Box B}$$

□-INTRODUCTION VIA SUBSUMPTION

$$\frac{x : [T]\, A \vdash M : B}{x : [\Box T]\, A \vdash M : \Box B}$$

2. Modal subtyping: □-elimination

$$f : \Diamond A \to B \iff f : A \to \Box B$$

2. Modal subtyping: $\Box$-elimination

$$f : \Diamond A \to B \iff f : A \to \Box B$$

Which suggests this elimination rule for $\Box$:

$$\frac{x : [T]\, A \vdash M : \Box B}{x : [\Diamond T]\, A \vdash \mathbf{unbox}\, M : B}$$

2. Modal subtyping: $\Box$-elimination

$$f : \Diamond A \to B \iff f : A \to \Box B$$

Which suggests this elimination rule for $\Box$:

$$\frac{x : [T]\, A \vdash M : \Box B}{x : [\Diamond T]\, A \vdash \textbf{unbox}\, M : B}$$

Happily, we can do this with subtyping!

SUBTYPING OUT OF $\Box$

$$\frac{[T]\, A <: B}{[T\Diamond]\, \Box A <: B}$$

$\Box$-ELIMINATION VIA SUBSUMPTION

$$\frac{x : [T]\, A \vdash M : \Box B}{x : [\Diamond T]\, A \vdash M : B}$$

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State of work

1. **Solved:** Pattern matching!
2. **Conjectured:** Substitution.
3. **Unsolved:** Internalizing $\Diamond$ w/o annotations.
4. **Open:** HM-style type inference?
5. Pfenning-Davies elim rule is derivable from substitution.

`http://www.rntz.net/files/tones.pdf`

`http://www.rntz.net/datafun`